

Dynamical localization and signatures of classical phase space

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Abstract

We study the dynamical localization of cold atoms in Fermi accelerator both in position space and in momentum space. We report the role of classical phase space in the development of dynamical localization phenomenon. We provide set of experimentally assessable parameters to perform this work in laboratory.

PACS numbers: 72.15.Rn, 47.52.+j, 03.75, 03.65.-w

I. INTRODUCTION

Existence of dynamical localization in a system is considered as signature of quantum chaology [1]. Rapid developments in atom optics [2–4] have made this subject a testing ground for the dynamical localization and hence for quantum chaology. Atomic dynamics in periodically driven systems, such as, an hydrogen atom in micro-wave field [5–7] an atom in modulated standing wave field [8–10], and the motion of an ion in Paul trap in presence of standing wave [11–13], have manifested the phenomenon of dynamical localization. Latest work on the dynamics of an atom in Fermi accelerator [14–16] has established the presence of dynamical localization in the system. In addition, this work has brought into light a new generic phenomenon of dynamical revivals of quantum chaology [15]. In this paper we study Fermi accelerator in atom optics domain and explain the role of classical chaology in the development of quantum dynamical localization.

II. ATOMIC FERMI ACCELERATOR

At the end of the first half of twentieth century, Enrico Fermi coined the idea that the origin and acceleration of cosmic rays is due to intragalactic giant moving magnetic fields [17]. Latter, Pustilnikov proved that a particle bouncing on an oscillating surface in gravitational field may get unbounded acceleration depending upon its initial location in phase space [18]. Based on this idea we have suggested Fermi accelerator for atoms in atom optics domain and have studied accelerating modes [15].

We may understand the atomic Fermi accelerator as: Consider a cloud of cesium atoms initially cold and stored in a magneto-optical trap. On switching off the trap, the atoms move along the \tilde{z} -direction under the influence of gravitational field and bounce off an atomic mirror [19,20]. The latter result from the total internal reflection of a laser beam incident on a glass prism. The incident laser beam passes through an acusto-optic modulator which provides a phase modulation to the evanescent field on the surface of glass prism [21]. This

model provides experimental realization of Fermi accelerator in the atom optics domain [14].

In order to avoid spontaneous emission we consider a large detuning between the laser light field and the atomic transition frequency. In presence of rotating wave approximation and dipole approximation the center-of-mass motion of the atom in ground state follows from the Hamiltonian

$$\tilde{H} \equiv \frac{p^2}{2m} + mg\tilde{z} + \frac{\hbar\Omega_{eff}}{4}e^{-2k\tilde{z}+\epsilon\sin\omega t} \quad (1)$$

Here, \tilde{p} is the momentum of the atom of mass m along the \tilde{z} -axis, g denotes the gravitational acceleration, Ω_{eff} is the effective Rabi frequency. The time dependent term expresses the spatial modulation of amplitude ϵ and frequency ω .

We introduce the dimensionless position and momentum coordinates $z \equiv \tilde{z}\omega^2/g$ and $p \equiv \tilde{p}\omega/(mg)$ and time $t \equiv \omega\tilde{t}$. Using these dimensionless coordinates we may express the Hamiltonian as

$$H \equiv \frac{p^2}{2} + z + V_0e^{-\kappa(z-\lambda\sin t)}, \quad (2)$$

where, we express the dimensionless intensity $V_0 \equiv \hbar\omega^2\Omega_{eff}/(4mg^2)$, steepness $\kappa \equiv 2kg/\omega^2$ and the modulation amplitude $\lambda \equiv \omega^2\epsilon/(2kg)$ of the evanescent wave. The commutation relation $[z, p] = [\tilde{z}, \tilde{p}]\omega^3/(mg^2) = i\hbar\omega^3/(mg^2)$ provides us the dimensionless Planck's constant $\hbar \equiv \hbar\omega^3/(mg^2)$.

The quantum dynamics of atom in Fermi accelerator [22] manifests dynamical localization in a certain localization window on modulation amplitude, $0.24 < \lambda < \sqrt{\hbar}/2$ [14,15]. The lower limit is obtained by Chirikov mapping and describes the onset of classical diffusion [23,24]. The upper limit of the localization window describes the phase transition of the quasi-energy spectrum of the Floquet operator from a point spectrum to a continuum spectrum [25–28]. Above this limit quantum diffusion sets in and destroys quantum localization. The conditions of classical and quantum diffusion, together, define the localization window.

III. DYNAMICAL LOCALIZATION VERSES THE CLASSICAL PHASE SPACE

In order to understand the effect of initial conditions on localization it is essential to understand how the classical phase space contributes towards the phenomenon of localization. We find that phase space structure of the system has direct effect on quantum evolution. In order to study quantum dynamics within the localization regime, we propagate an initial atomic wavepacket $\psi(z)$, expressed as

$$\psi(z) = \frac{1}{\sqrt{\sqrt{2\pi}\Delta z}} \exp\left(-\frac{(z-z_0)^2}{2\Delta z^2}\right) \exp\left(-i\frac{p_0 z}{\hbar}\right), \quad (3)$$

at $t = 0$, and propagate it in the atomic Fermi accelerator. Here, z_0 describes the average position, and p_0 denotes the average momentum of the wave packet. The widths of the wavepacket in position space and in momentum space are chosen such that they satisfy the minimum uncertainty condition.

We investigate the effect of classical resonances on dynamical localization in Fermi accelerator by propagating an atomic wavepacket from an initial height $z_0 = 20$, with initial momentum $p_0 = 0$. We select $\hbar = 1$ which provides localization window on modulation strength, λ , as $0.24 < \lambda < 0.5$. The initial widths of the wave packet are $\Delta z = 0.5$ in position space, and $\Delta p = 1$ in momentum space, corresponding to the minimum uncertainty parameters. We propagate the atomic wavepacket for a modulation amplitude of $\lambda = 0.4$ which lies well within the localization window. We note the probability distribution of the wavepacket in the atomic Fermi accelerator after evolution $t = 1000$, both in position space and in momentum space. We observe the classical phase space by means of Poincare' surface of section for the modulation strength $\lambda = 0.4$, as shown in Fig. 1.

So far as modulation strength is small, that is, $\lambda < \lambda_l = 0.24$, we have isolated resonances in classical phase space. In this domain, quantum dynamics mimics the classical dynamics and we do not find dynamical localization. The phenomenon of localization occurs after the overlap of resonances has occurred in the classical phase space, that is, above λ_l and persists until the quantum diffusion starts in the system [15].

Within the localization window we calculate the quantum mechanical position and momentum distributions of the atomic wavepacket. We compare our result with the classical phase space observed as Poincare' surface of section. We place the atomic wavepacket close to the second resonance. Therefore, we find that maximum probability density is localized there. Hence, a plateau structure occurs in the probability distribution both in position space and in momentum space which is at the second resonance of the phase space. Our numerical results show that the size of the plateau is equal to the size of the resonance. The tail of the initial Gaussian wavepacket falls exponentially into the phase space, therefore, it also occupies the other resonances but with the difference of orders of magnitude. The location of the first resonance is the closest to the second one, as we find from Poincare' section of the Fermi accelerator in Fig. 1. As a result a significant part of initially propagated atomic wavepacket lies in this region, and seems to be contributing to the plateau structure of second resonance. We observe that the next plateau corresponding to third primary resonance is approximately four orders of magnitude smaller, and the next to it corresponding to the forth resonance of phase space is further eight order of magnitude smaller. This helps us to infer that the atomic probability densities in position space and in momentum space localized into the regions of islands and the atomic wavepacket spreads over the stable island, making a plateau structure. Outside the stable islands the probability densities fall linearly into the stochastic sea.

Fishman *et. al.* [29] have suggested that the eigen functions decay as $\exp(-(n - \bar{n})/\ell)$ away from the mean level \bar{n} , in the momentum space. Therefore, we expect that the overall drop of probability distribution in momentum space is linear. In momentum space, our numerical experiment display the overall linear drop of probability distribution, whereas, in position space the probability distribution displays an overall drop according to square root law [14,15].

Hence within localization window the atomic wavepacket displays three interesting features: (i) *plateau structures* in regions corresponding to stable islands of phase space; (ii) *linear decay* in regions corresponding to stochastic sea; (iii) *overall decay* following square

root law in position space and following linear behavior in momentum space. This overall decay may be different for different systems.

We may understand this effect by relating the underlying energy spectrum of quantum dynamical system with the classical phase space. We [30] have tested that corresponding to a classical resonances there exist a local discrete spectrum, whereas, in the stochastic region we find quasi continuum. For the reason we observe that the probability distribution occupying local discrete spectrum of a resonance undergoes constructive interference and displays plateau structure, whereas the probability distribution occupying the quasi continuum spectrum undergoes destructive interferences and therefore falls linearly in phase space.

IV. DYNAMICAL LOCALIZATION AND CHANGING PLANCK'S CONSTANT

How the change in the effective Planck's constant effects the dynamical localization? In order to answer the question we propagate the atomic wavepacket in the Fermi accelerator considering the Planck's constant $\hbar = 4$ and compare it with $\hbar = 1$ case. We keep all the parameters the same as earlier. Following the previous procedure we note the probability distributions after an evolution time $t = 1000$ and display it in Fig. 2.

We note that the size of the initial minimum uncertainty wavepacket is larger due to the larger value of \hbar . Therefore, amount of the initial probability density falling into the stable islands becomes larger, as compared with the $\hbar = 1$ case, increasing the height of the plateaus.. Moreover, for larger \hbar the exponential tail of initial Gaussian wavepacket covers more resonances leading to more plateau structures in the localization arm. The size of the plateau corresponds to the size of resonances which are independent of the value of Planck's constant. As a result we conjecture that the size of the plateau remains the same for two different values of the Planck's constants.

Hence, for the larger value of the Planck's constant, we find that: (i) the size of the plateau in probability distributions is the same as it was for $\hbar = 1$; (ii) heights of the

plateaus are higher; *(iii)* more plateau structures are appearing.

Comparison between classical and quantum position distributions show that the plateau structures also appear in the classical cases. However, their height is larger than the corresponding quantum cases. In order to compare classical and quantum position distributions we propagate a classical ensemble. We note the classical distribution (solid thick line) in the Fermi accelerator after an evolution time $t = 1000$ and compare with the corresponding quantum mechanical distributions for $\bar{k} = 1$ (solid thin line) and for $\bar{k} = 4$ (dashed line).

The classical distributions in momentum space and in position space are entirely different from their quantum counterparts. The classical dynamics of the ensemble in Fermi accelerator model supports an overall quadratic distribution in momentum space supporting diffusive dynamics and linear distribution in position space [14,15]. Since the classical position and momentum distributions are the marginal integrations of phase space, we find that plateaus exist even in classical distributions. We find that the location and the size of the plateaus are the same in both classical and quantum cases, however, their heights differ. As compared to the corresponding classical counterparts, in the quantum mechanical case the heights of the plateaus is reduced, which may occur as a result of the dynamical tunneling of the probability to the other plateaus.

V. EXPERIMENTAL PARAMETERS

The reflection of atoms onto an evanescent wave mirror has been observed in many laboratory experiments [20,21,33]. In this section we connect our choice of parameters with the currently accessible technology and show that the effects we have predicted in this paper can be observed in a real experiment. We consider cesium atoms of mass $m = 2.21 \times 10^{-25}$ kg bouncing on the evanescent wave field with the decay length $k^{-1} = 0.455 \mu\text{m}$ and the effective Rabi frequency $\Omega_{eff} = 2\pi \times 5.9$ kHz. These parameters in presence of a modulation frequency of $\omega = 2\pi \times 1.477$ kHz, lead to $\bar{k} = 4$, $\kappa = 0.5$ and $V_0 = 4$. By choosing $\Omega_{eff} = 2\pi \times 14.9$ kHz, $k^{-1} = 1.148 \mu\text{m}$ and $\omega = 2\pi \times 0.93$ kHz we get $\bar{k} = 1$, keeping $\kappa = 0.5$ and $V_0 = 4$ which

are the values used in our calculations.

VI. ACKNOWLEDGMENT

We thank G. Alber, I. Bialynicki-Birula, M. Fortunato, M. El. Ghafar, R. Grimm, B. Mirbach, M. G. Raizen, V. Savichev, W. P. Schleich, F. Steiner and A. Zeiler for many fruitful discussions.

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FIGURES

FIG. 1. A comparison between classical phase space and quantum mechanical distributions: At the top we show the Poincarè surface of section for a modulation strength $\lambda = 0.4 > \lambda_l$. In (a) we display the quantum mechanical momentum and in (b) the corresponding position distribution after a propagation time $t = 1000$ for $\hbar = 1$, using the same value of modulation strength, on logarithmic scales. Comparing the probability distributions with the classical phase space we clearly find the probability confinement in the region of a resonance. However, the probability distribution decays exponentially into the stochastic region.

FIG. 2. Change in the position probability distribution with increasing Planck's constant: We display the probability distribution in position space for $\hbar = 4$. All the other parameters are the same as in Fig. 1. We find that with increasing the effective Planck's constant the height of the plateaus rises indicating an increase in the probability distribution. However, their location and size remain approximately the same due to the fixed size of the resonance area.

FIG. 3. Plateau structures in the classical and in the quantum mechanical position distributions: We observe that the plateau structures also exist in the classical distribution. Their location and size are the same as in quantum distributions, however, they exhibit larger heights. This implies that, in the classical case, the trapped probability distribution is larger as compared to the corresponding quantum distribution. A decay of probability in quantum case may result due to dynamical quantum tunneling which appears only in quantum cases. In the classical case we propagated 60000 atoms and noted their distribution after $t = 1000$, whereas, in our quantum calculation we followed the same procedure as in Fig. 1





